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**OPTIMUM DESIGN OF STRUCTURES SUBJECT TO  
GENERAL PERIODIC LOADS**

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## INTRODUCTION

The optimum design of structures subject to a single harmonic load has been considered by Icerman [1] and Plaut [2]. Icerman [1] studied a rod acted upon by a single harmonic tip load, and he minimized the amplitude of the tip's steady state displacement subject to fixed volume. His derivation of the necessary optimality criterion required a steady state extension to the principle of minimum potential energy. This new extremum principle can be used to establish the global sufficiency of the optimality condition, provided sufficiently restrictive conditions are placed upon the admissible designs.

Plaut [2] generalized Icerman's work by minimizing the amplitude of the steady state deflection at any specified location of the structure. His problem allowed for several harmonic loads provided all were driven at the same frequency. He derived the necessary optimality condition by first extending the principle of mutual stationary potential energy to the steady state. He did not address whether the optimality condition was also sufficient for the optimal design.

This paper initially addresses a simplified version of Icerman's problem. The nature of the restrictive conditions that must be placed on the design space in order to ensure an analytic optimum are discussed in detail. Icerman's problem is then extended to include multiple forcing functions with different driving frequencies. And the conditions that now must be placed upon the design space to ensure an analytic optimum are again discussed. An important finding is that all solutions to the optimality condition (analytic stationary design) are local optima, but the global optimum may well be non-analytic.

The more general problem of distributing the fixed mass of a linear elastic structure subject to general periodic loads in order to minimize some measure of the steady state deflection is also considered. This response is explicitly expressed in terms of Green's functional and the abstract operators defining the structure. The optimality criterion is derived by differentiating the response with respect to the design parameters [3]. The theory is applicable to finite element as well as distributed parameter models.

## AN ELEMENTARY FORCED VIBRATION

Consider a rod of length  $L$  whose cross-sectional area  $S$  is piecewise constant. In the simplest case  $S = S_1$  for  $0 \leq x < L/2$  and  $S = S_2$  for  $L/2 < x \leq L$  (Fig. 1). The design ratio  $S_1/S_2$  is denoted by  $R$ . The tip load is  $F \cos \omega_D t$ . The steady state response can be represented by  $U(x) \cos \omega_D t$ , where  $|U(L)|$  is the amplitude of the tip response. Figures 2 and 3 show, respectively, the natural frequency of the design and tip amplitude as a function of the design ratio  $R$ . For designs with  $R = R^*$ , the design and driving frequencies coincide and hence the resonance condition  $^{**}$  shown, Icerman's [1] optimality condition is sufficient provided only designs  $R > R^*$  are considered.

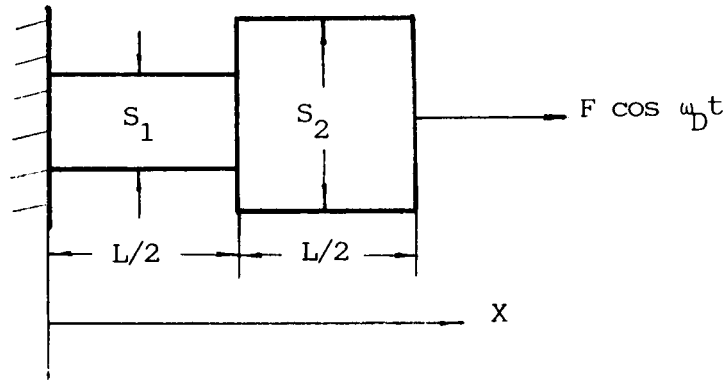


Figure 1. Two-Segment Rod Subject To A Single Driving Frequency

# FREQUENCY AND TIP-DEFLECTION AMPLITUDE VS DESIGN

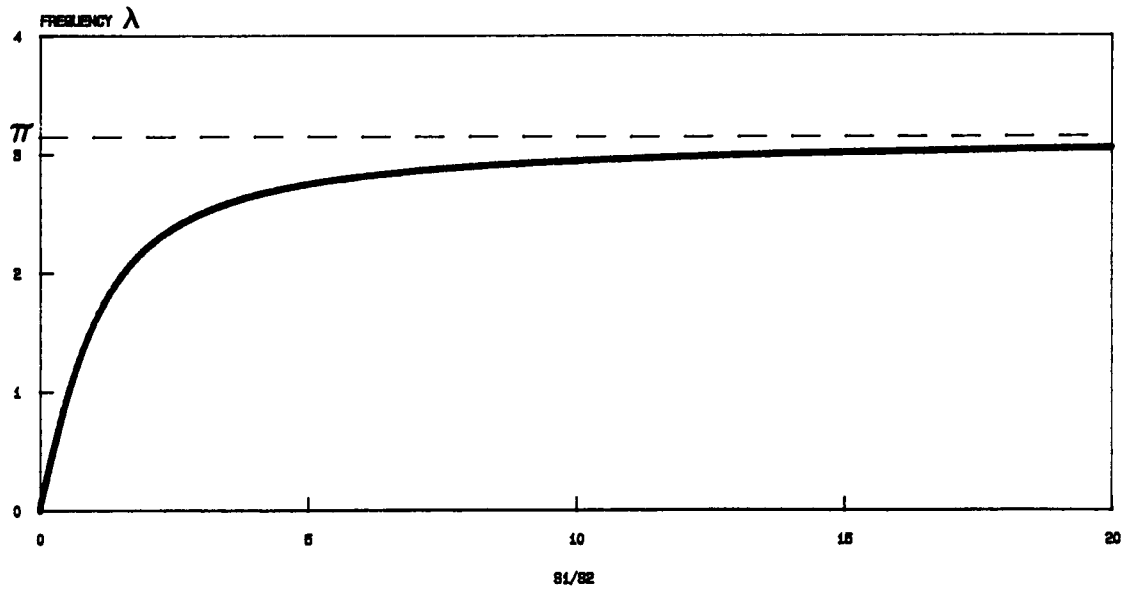


Fig. 2. Non-Dimensional Frequency vs. Design. The non-dimensional frequency  $\lambda$  is defined as  $\omega L \sqrt{\rho/E}$  where  $\rho$  is mass per unit length and  $E$  is Young's modulus.

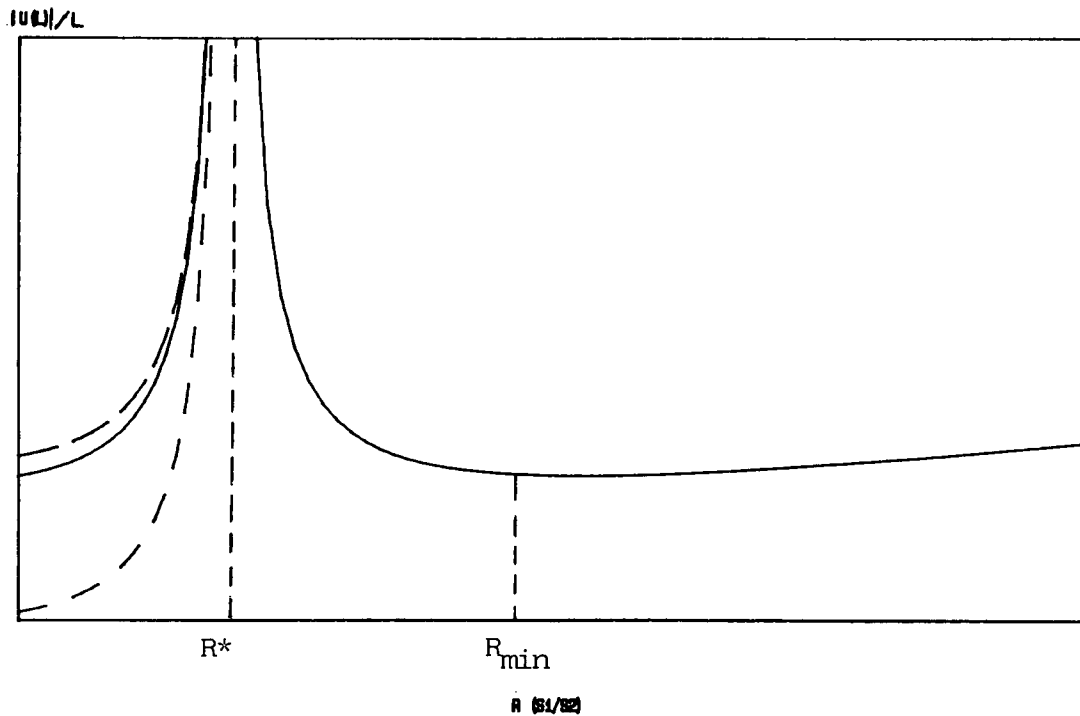


Fig. 3. Tip Deflection vs. Design.  $R_{min}$  is the analytic local optimum. Depending upon the value of  $U(0)$ ,  $R_{min}$  may or may not be globally the optimum.

## TWO ELEMENT ROD SUBJECTED TO TWO DRIVING FREQUENCIES

The rod shown in Figure 1 is now subjected to a tip load consisting of two frequencies -  $F=F_1 \cos \omega_D t + F_2 \cos 2 \omega_D t$ . While the addition of the second driving frequency does not affect the period of the response, it does add a minor complication to the calculation of the maximum response. The response can be expressed in the form  $U_1(x) \cos \omega_D t + U_2(x) \cos 2 \omega_D t$ . The maximum tip response  $U_{\max}$  is obtained by setting  $x=L$  and maximizing the magnitude of the response over one period.

There are two resonant frequencies and, in general, two designs  $R_1^*$  and  $R_2^*$  for which the fundamental frequency equals one of the resonant frequencies. A typical response is depicted in Figure 4. Clearly, there are up to three local extrema; all are minima and one is not analytic. Analogous to the simpler case of only one forcing frequency, one or both of the resonant designs may not exist.

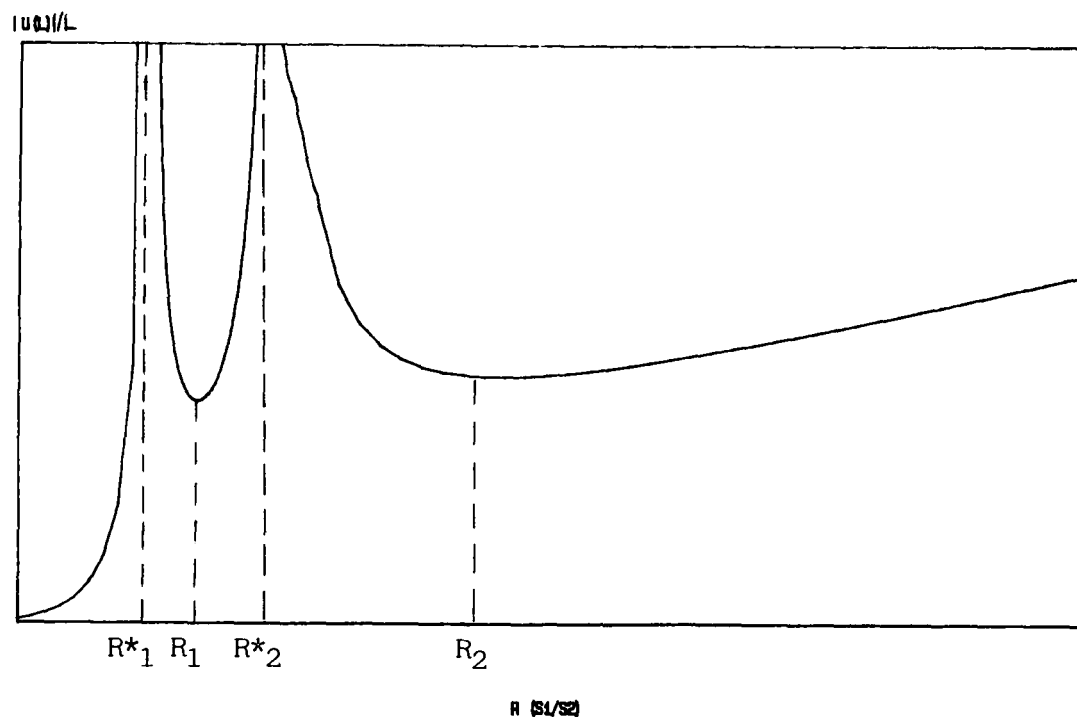


Figure 4. Typical Maximum Response as a Function of the Design Parameter.  $R_1$  and  $R_2$  are analytic local optimum designs.

# OPTIMIZATION USING GREEN'S FUNCTIONAL

Consider the following boundary value problem

$$(T^* E T + \alpha M)u = f \text{ in } \Omega$$

$$B \gamma u = g \text{ in } \partial \Omega_1$$

$$B^* \gamma^* E T u = h \text{ in } \partial \Omega_2$$

where  $T, T^* \dots L_2$  adjoint operators

$E(S) \dots$  Linear stiffness operator

$M(S) \dots$  Linear mass operator

$\alpha \dots \dots$  A scalar

$\gamma, \gamma^* \dots$  Trace operators mapping functions defined in  $\Omega$  onto functions defined on  $\partial \Omega_1$  and  $\partial \Omega_2$ , respectively.

$B, B^* \dots$  Boundary operators

$\partial \Omega_1, \partial \Omega_2 \dots$  Complementary subsets of  $\partial \Omega$

$S \dots \dots$  Design variable(s)

The following integration by parts formula, due to Oden and Reddy [4], is postulated

$$\begin{aligned} (Tu, ETv)_\Omega - (u, T^* ETv)_\Omega \\ = (\gamma u, B^* \gamma^* ETv)_{\partial \Omega_2} - (B \gamma u, \gamma^* ETv)_{\partial \Omega_2} \end{aligned}$$

Also  $E, M$  satisfy

$$(u, Ev)_\Omega = (v, Eu)_\Omega$$

$$(u, Mv)_\Omega = (v, Mu)_\Omega \text{ for every admissible } u \text{ and } v.$$

# STRUCTURAL DYNAMICS

The equations governing linear structural dynamics can be expressed as

$$T^* ET u + M \ddot{u} = f \quad \text{in } \Omega$$

$$B \gamma u = g \quad \text{on } \partial\Omega_1$$

$$B^* \gamma^* ET u = h \quad \text{on } \partial\Omega_2$$

If  $f, g$ , and  $h$  all have the same periodicity,

$$f = \sum_{n=1}^N f_n(x) \cos n \omega_D t$$

$$g = \sum_{n=1}^N g_n(x) \cos n \omega_D t$$

$$h = \sum_{n=1}^N h_n(x) \cos n \omega_D t$$

The solution  $u(x, t)$  satisfies

$$u = \sum_{n=1}^N U_n(x) \cos n \omega_D t$$

The Fourier coefficients satisfy

$$L_n \equiv (T^* ET - n^2 \omega_D^2 M) U_n = f_n \quad \text{in } \Omega$$

$$B \gamma U_n = g_n \quad \text{on } \partial\Omega_1$$

$$B^* \gamma^* ET U_n = h_n \quad \text{on } \partial\Omega_2$$

It can be shown that the solution is

$$U_n = (f_n, G_n)_\Omega + (g_n, \gamma^* ET G_n)_{\partial\Omega_1} + (h_n, \gamma G_n)_{\partial\Omega_2}$$

where  $G_n$  is the Green's function corresponding to the operator  $L_n$ .

# SOLUTION TO VARIATIONAL EQUATIONS

Define

$$\delta U_n = U_n(x, S+\delta S) - U_n(x, S)$$

Then, by using Oden and Reddy's integration by parts formula and taking the first variation of the differential equation specifying  $U_n$ , it can be shown that

$$\delta U_n = -(\delta E T U_n, T G_n)_\Omega + n^2 \omega_D^2 (\delta M U_n, G_n)_\Omega$$

Consequently,

$$\delta u(x, t) = \sum_1^N [n^2 \omega_D^2 (\delta M U_n, G_n)_\Omega - (\delta E T U_n, T G_n)_\Omega] \cos n \omega_D t$$

To determine whether any particular design which causes a stationary response at fixed  $x$  and  $t$  also minimizes the response can be obtained by variational calculus. The second variation  $\delta^2 u$  clearly involves the terms  $\delta^2 E$ ,  $\delta U_n$ ,  $\delta^2 M$ . However,  $\delta U_n$  is determined above, and

$$\delta G_n = - (T G_n \delta E T G_n)_\Omega + n^2 \omega_D^2 (\delta M G_n, G_n)_\Omega$$

Consequently,  $\delta^2 u$  is completely determined by the variations in the operators  $E$  and  $M$ .



# APPLICATIONS TO A ROD

A rod with varying cross-sectional area  $A(x)$  is fixed at  $x=0$  and acted upon by the horizontal periodic force

$$\sum F_n \cos n \omega_D t$$

at the tip  $x=L$ . The steady state equations are

$$-(E_0 A U_n')' - c^{-2} E_0 A \omega_D^2 n^2 U_n = F_n \delta(L-x)$$

$$U_n(0) = E_0 A U_n'(L) = 0$$

where  $E_0$  is Young's modulus and  $c^{-2} = \rho/E$ . Clearly

$$U_n(x) = F_n G_n(L, x)$$

Upon making the following identification for the operators:

$$T = -T^* = d/dx, \quad E = E_0 A = S(x), \quad M = \rho A = c^{-2} S(x)$$

the solution for  $\delta U_n(L)$  becomes

$$\delta U_n(L) = \frac{1}{F_n} \left\{ \int_0^L \left\{ U_n'^2 - n^2 \omega_D^2 c^{-2} U_n^2 \right\} \delta S dx \right.$$

Now, let  $T$  be selected so that

$$u(L, T) \equiv U_{\max}$$

By imposing the fixed mass constraint

$$\int_0^L \delta S dx = 0,$$

the optimality condition for minimizing  $U_{\max}$  over all admissible designs  $S$  is obtained, i.e.

$$\sum_{n=1}^N \frac{\cos n \omega_D T}{F_n} (U_n'^2 - n^2 \omega_D^2 c^{-2} U_n^2) = \lambda$$

For rods consisting of piecewise constant cross-sections  $S_i$  in the domain  $x_{i-1} < x < x_i$ , the optimality condition takes the form

$$\sum_{n=1}^N \frac{\cos n \omega_D T}{F_n} \int_{x_{i-1}}^{x_i} (U_{ni}'^2 - n^2 \omega_D^2 c^{-2} U_{ni}^2) dx = \lambda_i$$

# RESULTS FOR A TWO-SEGMENT ROD WITH ONE DRIVING FREQUENCY

Table 1 shows results for a single driving. The non-dimensional frequency  $\Omega$  is  $\omega_D L/c$ , the analytic optimum design stiffness ratio  $R$  is  $S_1/S_2$ ,  $U_{opt}$  is the analytic minimum,  $U_{unif}$  is the amplitude of vibration for  $R=1$  and  $U_o$  is the non-analytic minimum obtained by letting  $R$  approach zero.

It is observed that for low frequencies, the analytic optimum is clearly superior to any other design. For  $\Omega > 0.854$ , the awkward non-analytic optimum provides a smaller amplitude for the response than does the analytic design. For still larger frequencies  $\Omega > 1.743$ , even the uniform design is preferable to the analytic local optimum.

TABLE 1. Comparison of Analytic Optimum to Non-Analytic Optimum and Uniform Designs.

$\Omega$	$R$	$U_{opt}/U_{unif}$	$U_{opt}/U_o$
0.1	1.005	0.99999	0.010
0.3	1.046	0.999	0.093
0.5	1.130	0.996	0.278
0.7	1.266	0.982	0.604
0.854	1.415	0.957	1.000
1.0	1.597	0.911	1.625
1.4	2.419	0.497	4.066
1.743	3.828	1.000	13.657
1.8	4.176	2.824	19.734

# RESULTS FOR A TWO-SEGMENT ROD WITH TWO DRIVING FREQUENCIES

Table 2 shows numerical results for two driving frequencies. The symbols  $\Omega$  and  $R$  are defined the same as in the previous example. Here,  $U_{opt}$ ,  $U_{unif}$  and  $U_o$  are, respectively, the maximum absolute value of the displacement over one period for a local analytical optimal design, uniform design, and a rod for which  $S_1$  tends to zero. The forcing function is  $F_1 \cos \omega_D t + F_2 \cos 2\omega_D t$ . The non-dimensional frequency  $\Omega_D$  is fixed at 0.6 for this example. Note that there are two analytic local optimal designs  $R_1$  and  $R_2$ . In the third and fourth columns in Table 2 are comparisons based upon the design  $R_1$  while the sixth and seventh columns are comparisons based upon  $R_2$ .

The global optimum is obtained by searching the fourth and seventh columns. Since the entries in Column 4 always exceed the corresponding entry in Column 7, it follows that  $R_2$  always provides a better solution than  $R_1$ . Further, for  $F_1 \geq F_2$ ,  $R_2$  is the global optimum but for  $F_1 < F_2$ ,  $R=0$  is the global optimum.

TABLE 2. Comparison of Both Analytic Optimal Designs with Non-Analytic Optimal Designs with Non-Analytic Optimum and Uniform Designs. All data is for  $\Omega_D = 0.6$

$F_1/F_2$	$R_1$	$\frac{U_{opt}}{U_{unif}}$	$\frac{U_{opt}}{U_o}$	$R_2$	$\frac{U_{opt}}{U_{unif}}$	$\frac{U_{opt}}{U_o}$
1/9	0.168	0.81	2.02	1.89	0.80	1.99
2/8	0.194	0.99	1.88	1.85	0.81	1.54
3/7	0.213	1.15	1.72	1.79	0.83	1.24
4/6	0.234	1.29	1.56	1.74	0.85	1.03
5/5	0.250	1.42	1.41	1.68	0.87	0.87
6/4	0.270	1.53	1.27	1.61	0.89	0.74
7/3	0.288	1.63	1.13	1.54	0.92	0.64
8/2	0.313	1.69	1.00	1.45	0.94	0.55
9/1	0.347	1.69	0.85	1.34	0.97	0.48

# ADDITIONAL RESULTS FOR TWO DRIVING FREQUENCIES

The identical problem as the preceding one is considered. Here, however, results are presented for fixed force amplitudes where  $F_1 = F_2$ . The driving frequency  $\Omega_D$  is varied in Table 3.

Of the analytic solutions, it is clear that  $R_1$  is superior to  $R_2$  for values of  $\Omega_D$  closer to 1.0, but  $R_2$  is better for the smaller driving frequencies. However  $R_1$  never provides the optimum global design. For forcing frequencies  $\Omega_D < 0.6$ ,  $R_2$  is the global optimum; otherwise the global optimum is not analytic.

TABLE 3. Comparison of Both Analytic Optimal Designs with Non-Analytic Optimum and Uniform Designs. All data is for  $F_1/F_2 = 1$ .

$\Omega_D$	$R_1$	$\frac{U_{opt}}{U_{unif}}$	$\frac{U_{opt}}{U_o}$	$R_2$	$\frac{U_{opt}}{U_{unif}}$	$\frac{U_{opt}}{U_o}$
0.2	0.025	16.6	1.10	1.05	0.999	0.07
0.4	0.105	3.93	1.27	1.24	0.985	0.30
0.6	0.251	1.42	1.40	1.68	0.87	0.87
0.8	0.482	0.14	1.76	2.67	0.18	2.31
1.0	0.824	0.96	2.36	5.09	2.84	7.00

# TWO SEGMENT ROD SUBJECT TO THREE FREQUENCIES

The two segment rod is now subjected to the tip force  $F_1 \cos \omega_D t + F_2 \cos 2 \omega_D t + F_3 \cos 3 \omega_D t$ . There are now three resonant designs and three local optimal designs in addition to the non-analytic optimal design as  $R \rightarrow 0$ . Table 4a contains results for the lowest driving frequency  $\Omega_D = 0.3$ . In all cases  $R_3$  provides the global optimum. However, if  $\Omega_D = 0.6$  (Table 4b), then  $R_2$  is the global optimum.

TABLE 4. Comparison of Three Analytic Local Optimum Designs with Non-Analytic Local Optimum Design.  $\Omega_D = 0.3$  for (4a) and  $\Omega_D = 0.6$  for (4b),

$F_2/F_1$	$F_3/F_1$	$R_1$	$\frac{U_{opt}}{U_o}$	$R_2$	$\frac{U_{opt}}{U_o}$	$R_3$	$\frac{U_{opt}}{U_o}$
0.50	0.50	0.065	0.82	0.157	0.40	1.22	0.18
0.75	0.75	0.061	0.89	0.158	0.52	1.25	0.21
1.00	1.00	0.058	0.94	0.158	0.54	1.27	0.34

(4a)

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0.50	0.50	0.286	1.07	0.825	0.67	3.12	1.38
0.75	0.75	0.267	1.15	0.830	0.76	3.24	1.72
1.00	1.00	0.254	1.20	0.838	0.95	3.32	2.02

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(4b)

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